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Some Remarks Concerning Nonnegative Harmonic Functions

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1. We treat questions concerning nonnegative harmonic functions on the upper half-plane, $U = \{\operatorname{Im} z > 0\}$. With the aid of Kjellberg's lemma [4] a truncation convergence theorem (Theorem 4.1) is obtained. A corollary (Theorem 4.2) of this theorem leads very simply to Loomis's converse [5] of the (sectorial limit) Fatou theorem for nonnegative harmonic functions on U. For important extensions in several directions of Loomis's work reference is made to the papers of Gehring [1–3] bearing on the subject.

2. Kjellberg's Lemma. This useful lemma may be stated as follows:

LEMMA. *Let u, u_1, u_2 be nonnegative harmonic functions on a region A satisfying $u \leqslant u_1 + u_2$. Then there exist nonnegative harmonic functions v_1, v_2 on A such that $v_i \leqslant u_i$, $i = 1, 2$, and $u = v_1 + v_2$.*

Proof. Given a subharmonic function $w(\not\equiv -\infty)$ on A possessing a harmonic majorant, we let Mw denote the least harmonic majorant of w. Turning to the situation of the lemma we note that

$$(u - u_2)^+ \leqslant u, u_1 .$$

Hence

$$0 \leqslant M(u - u_2)^+ \leqslant u, u_1 ,$$

so that

$$0 \leqslant u - M(u - u_2)^+ \leqslant u_2 .$$

The proof is completed on noting that $v_1 = M(u - u_2)^+$ and $v_2 = u - v_1$ serve.

3. The Poisson–Stieltjes representation for nonnegative harmonic functions on U. This standard result may be formulated as follows:

Given u, nonnegative harmonic on U, there exists a unique pair (p, α), where p is a nonnegative real number and α is a monotone nondecreasing *function on the real line* R *satisfying* $\alpha(0) = 0$, $\alpha(t) = [\alpha(t+) + \alpha(t-)]/2$, $t \in R$, *such that*

$$u(z) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \operatorname{Im}(t-z)^{-1}\, d\alpha(t) + p \operatorname{Im} z, \qquad z \in U. \tag{3.1}$$

4. The convergence theorem. Let $(u_n)_1^\infty$ be a sequence of nonnegative harmonic functions on U tending pointwise to u (finite-valued). Let α be associated with u in the sense of Section 3 and let α_n be correspondingly associated with u_n. Let a, b satisfy $-\infty < a < b < +\infty$ and be such that α is continuous at both a and b. We introduce v_n, v by

$$v_n(z) = \frac{1}{\pi} \int_a^b \operatorname{Im}(t-z)^{-1}\, d\alpha_n(t),$$

and

$$v(z) = \frac{1}{\pi} \int_a^b \operatorname{Im}(t-z)^{-1}\, d\alpha(t),$$

$z \in U$. They are, of course, nonnegative harmonic on U. We show

THEOREM 4.1. *(v_n) tends to v, uniformly on compact subsets of U.*

[It is to be noted that the multiples of $\operatorname{Im} z$ appearing in the representations of u_n and u are without effect as far as the theorem is concerned.]

Proof. We show: *If (v_n) is pointwise convergent, then* $\lim v_n = v$. It is to be observed that here pointwise convergence implies uniform convergence on compact subsets of U. The pointwise convergence of (v_n) may be referred to the fact that the asserted limit property is valid when (v_n) is replaced by a pointwise convergent subsequence. Let $w = \lim v_n$. At all events, $w \leqslant u$. Further, w vanishes continuously at each point of the frontier of U (in the sense of the topology of the extended plane) not in the segment $[a, b]$.

On replacing $[a, b]$ by a slightly larger segment $[A, B]$, $A < a < b < B$, we see, thanks to the Poisson–Stieltjes representation for u, that with h_1 given by

$$h_1(z) = \frac{1}{\pi} \int_A^B \operatorname{Im}(t-z)^{-1}\, d\alpha(t), \qquad z \in U, \tag{4.1}$$

and $h_2 = u - h_1$, h_1 and h_2 are nonnegative harmonic on U. We apply the Kjellberg lemma to w, h_1, h_2. On noting that

$$\lim_\zeta \min\{w, h_2\} = 0$$

for each ζ in the frontier of U (as above), we conclude by the boundary maximum principle that the second term in the Kjellberg representation of w is 0. Hence $w \leqslant h_1$. By the continuity of α at a and b, we obtain $w \leqslant v$.

To obtain inequality in the opposite sense, we proceed as follows. We now take A and B so that $a < A < B < b$. Since $u - w = \lim(u_n - v_n)$ and each of the functions $u_n - v_n$ is nonnegative harmonic on U and has limit 0 at each point of the interval (a, b), it follows (from the well-known inequalities, paralleling the Harnack inequalities, for nonnegative harmonic functions on a semicircular disk, which vanish continuously on the diameter) that $u - w$ vanishes continuously at each point of the interval. On taking h_1 of (4.1) with the present A and B and applying the boundary maximum principle to

$$[(u - w) + h_1] - u,$$

we conclude that $h_1 \leqslant w$. By the continuity of α at a and b, it follows that $v \leqslant w$.

Hence $v = w$ and the proof of the theorem can now be rapidly completed with the aid of the observations of the first paragraph of the proof.

Let P denote the set of nonnegative harmonic functions on U. Given a, b satisfying $-\infty < a < b < +\infty$, we consider the map T of P into itself which assigns to $u \in P$ the function

$$z \to \frac{1}{\pi} \int_a^b \operatorname{Im}(t - z)^{-1}\, d\alpha_u(t), \qquad z \in U,$$

where α_u is the monotone function appearing in the Poisson–Stieltjes representation of u. Understanding that the topology on P is that of uniform convergence on compact subsets of U, we see that T is continuous at each $u \in P$ for which α_u is continuous at a and b, thanks to Theorem 4.1 and the local countable base property of the topology on P.

We are led to the following theorem.

THEOREM 4.2. *Suppose that α_{u_0} is continuous at a and b. Then*

$$u \to \alpha_u(b) - \alpha_u(a)$$

is continuous at u_0.

Let l_u denote the Radon measure on $C[a, b]$, the space of real-valued continuous functions on the segment $[a, b]$, defined by

$$l_u(f) = \int_a^b f\, d\alpha_u .$$

Then the map $u \to l_u$ is continuous at u_0, the range topology being the weak topology of $C^*[a, b]$, the conjugate space of $C[a, b]$.*

Proof. The first assertion follows on considering the expansion about ∞ of Tu extended by Schwarzian reflexion to the complement of $[a, b]$ with respect to the complex plane and on observing thereupon the continuous dependence of the coefficient of $\operatorname{Im}(z^{-1})$ of this expansion upon u. Account is to be taken of the fact that for u sufficiently near u_0 the extended Tu are uniformly bounded in some punctured neighborhood of ∞.

The second assertion may be established by applying the first assertion to a and x (replacing b), $a < x \leqslant b$, such that α_{u_0} is continuous at x, and thereupon using approximating sums for entering Riemann–Stieltjes integrals with points of subdivision taken as points of continuity of α_{u_0}.

5. Loomis's Converse Theorem. This theorem may be formulated as follows:

THEOREM. *Let $u \in P$ possess the finite sectorial limit c at 0. Then $\alpha_u'(0) = c$.*

Proof. We define $\varphi(s)$, $0 \leqslant s < +\infty$, as follows: $\varphi(0)$ is the constant c on U and $\varphi(s)$ is $z \to u(sz)$, $z \in U$, when $0 < s < +\infty$. Then φ is a continuous map of its domain into P. We observe that

$$\alpha_{\varphi(0)}(t) = ct$$

and

$$\alpha_{\varphi(s)}(t) = \alpha_u(st)/s, \qquad 0 < s < +\infty.$$

We apply the first assertion of Theorem 4.2, $\varphi(0)$ taking over the role of u_0. With $a = 0$ and $b = 1$ we obtain

$$\lim_{s \downarrow 0} \alpha_u(s)/s = c,$$

while with $a = -1$ and $b = 0$ we obtain

$$\lim_{s \downarrow 0} \alpha_u(-s)/s = c.$$

The theorem follows.

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